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Non-Markovian autoresonant dynamics of tunneling from discrete to continuum modes

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We study the autoresonant dynamics of a discrete level coupled to a continuum, and show that passing adiabatically through a linear resonance, above a well-defined threshold, yields a transition to nonlinear phase locking and linear non-Markovian decay to the continuum. This process results in broadening of the population of the continuum modes beyond its natural linewidth. This concept can be employed to alter spontaneous emission, where driving an atom into phase locking with continuum modes will yield the emission of short pulses.

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I. INTRODUCTION

Efficiently driving a nonlinear oscillator by an external oscillating force is not a trivial task. At low amplitudes, the oscillator can be captured into resonance with external oscillations. But, as the amplitude of nonlinear oscillations increases, the resonance frequency varies, and the nonlinear oscillator becomes detuned from the source, leading to asynchrony in their phases. Without phase synchronization, the power transfer process is inefficient: the power flows periodically from the source to the oscillator and back. An efficient technique for overcoming these difficulties exploits autoresonance: a nonlinear phenomenon in which a driven oscillator is captured into a continuous resonance with an external oscillator, despite dynamic variations in the system parameters. The phenomenon works as follows. A driver, coupled to a nonlinear oscillator, oscillates with a chirped, slowly varying, time-dependent frequency. When the frequency passes the resonant frequency of the driven oscillator, their phases lock, and remain locked continuously. The phase locking is maintained by a continuous increase in the amplitude of the driven oscillator, resulting in efficient amplification. Since the main requirement for such dynamics is just the presence of a nonlinearity, autoresonance is a very general phenomenon relevant to many physical areas, ranging from plasma [1] and fluid dynamics [2] to atom physics [3], Bose-Einstein condensates (BECs) [4], superconducting Josephson junctions [5,6], optical waves [7,8], and even planetary dynamics [9].

Traditionally, autoresonance has been studied in systems that couple a driver to single a driven system, although some works also studied the coupling between a driver and a discrete set of driven systems, such as the excitation of multiphase waves [10] or the simultaneous amplification of incoherently coupled optical waves [11]. Interestingly, autoresonance coupling between an oscillator and a continuous set of oscillators (a continuum) has never been studied. In a broad perspective, coupling between an oscillator and a continuum of modes describes a variety of fundamental phenomena, such as spontaneous emission from excited atoms, radioactive decay (tunneling), resonance states (leaky modes), etc. All of these phenomena are described by a discrete level coupled to a broadband continuum (a continuous set of modes). The general outcome of such processes is Markovian dynamics, in which the discrete level decays exponentially into the continuum. Even if the discrete level is nonlinear and chirped, as long as the continuum is wide enough and has no singularities, memory effects will be negligible, and the decay will be exponential, that is, a Markovian process. However, when the continuum is not broadband anymore, or if it has singularities, the dynamics can become non-Markovian, where memory effects play major role in the evolution of the system, and the decay deviates from being exponential, and can even be inhibited altogether. For example, an atom spontaneously emitting a photon near a band edge of a photonic crystal decays in a nonexponential fashion, due to the singularity in the density of states near the band edge [12]. Another method for non-Markovian dynamics of a quantum system coupled to a reservoir relies on frequent measurements or modulation of the coupling constant, which also yields nonexponential decay [13] and controlled decoherence even in the presence of Kerr nonlinearity [14]. These effects, known as the quantum Zeno and anti-Zeno effects, can lead to acceleration or deceleration of decay process of the level into the continuum. These findings raise the following interesting question. Consider a discrete level, coupled to a continuum with a bandwidth comparable to the other relevant frequencies (chirp rate, nonlinearity response time). Is it possible to phase lock the discrete level with the continuum modes? If indeed this is possible, then what would be the phase to which the discrete level locks? Does the discrete level lock with all the continuum modes, or with just a fraction of the continuum? Is there a threshold mechanism for the process?

In this paper, we study the dynamics of a discrete level coupled to a continuum of modes. The discrete level is nonlinear and its frequency is chirped. The continuum is narrow enough to prevent Markovian dynamics and is also nonlinear. Initially, the frequency of the discrete level is strongly detuned from the peak of the continuum such that it barely couples to it. As time evolves, the frequency of the discrete level approaches the center of the continuum, and thus it starts to decay into the continuum. Below a specific threshold, we find that the decay is arrested, and the discrete state remains excited while the system passes the continuum. On the other hand, above that threshold, the process involves simultaneous phase locking of the discrete state with the entire continuum of modes, which consequently results in a linear decay (rather than exponential) into the continuum, and broadening the distribution of the continuum modes. In fact, the distribution of the eventually populated continuum modes has a width larger than the natural width of the continuum.

II. NONLINEAR NON-MARKOVIAN COUPLED MODE THEORY

We begin with the coupled equations describing the dynamics of the discrete system coupled to the continuum in the presence of the Kerr nonlinearity [11,14]:

$$\partial_t b = -i \sum_k g_k a_k - i\chi_1 |b|^2 b - i\alpha t b - i\omega_0 b, \quad \text{(1a)}$$

$$\partial_t a_k = -i g_k b - i\chi_2 \sum_{k'} |a_{k'}|^2 a_k - i\omega_k a_k. \quad \text{(1b)}$$

Here, $a_k(t)$ is the time-dependent amplitude of the kth mode of the continuum with frequency ω_k, and $b(t)$ is the amplitude of the discrete system with the resonance frequency ω_0. χ_1 and χ_2 are the strengths of the Kerr nonlinearity of the discrete state and of the continuum, respectively. For simplicity, we assume that χ_2 is independent of k; that is, the strength of the nonlinearity is the same for all the continuum modes. The chirp is achieved by adiabatically varying the resonance frequency of the discrete system, with a rate α. Henceforth, we use positive chirp $\alpha > 0$, although the phenomena are generic and could be observed not only with a linear chirp but also with other types of potential variations. Also, g_k is the coupling constant between the discrete level and the kth mode, where we assume that g_k is a real function of k. For concreteness, we model the continuum as a Lorentzian, centered at ω_0 with a frequency width of Γ, and coupling strength κ, that is,

$$g_k^2 = \kappa^2 \frac{\Gamma}{\pi[c(k-k_0)^2 + (\Gamma/2\pi)^2]} \frac{c}{L}. \quad (2)$$

The dispersion relation relates the wave vector and the frequency, $\omega = ck$. c is the relevant velocity and k_0 is the wave vector corresponding to the central frequency ω_0. L is the length of a box containing the continuum modes (which is taken to be infinite). Such a continuum describes, for example, the coupling between an atom and the continuum of electromagnetic vacuum modes in a lossy cavity. The atom in such a cavity is coupled to a narrow band of electromagnetic continuum modes, due to the presence of the cavity (in contrast to a free atom that is coupled to a broadband continuum). As a result, the spontaneous emission decay rate in the cavity is smaller than the decay rate of a free atom (the Purcell effect). The decay of the excited state of the atom gives rise to population of the radiation vacuum modes at the expense of the amplitude of the discrete state (the excited state), exactly like the system described here.

First, to gain some intuition about the system, we study the initial stages of evolution in the system. We assume that initially only the discrete state is occupied, that is, $b(t_0) = 1$ at $t_0 < 0$. We also assume that the amplitude of b varies only slightly with time, and that at early enough time, the population of the continuum modes is negligible ($a_k(t) \approx 0$ for every k)

$$b(t) \approx \exp\left[-i\alpha\left(t^2 - t_0^2\right)/2 - i\omega_0(t-t_0) - i\chi_1 \int_{t_0}^{t} |b(t')^2| dt'\right]. \quad (3)$$

Next, we define an effective nonlinearity strength $\chi = \chi_1 + \chi_2$ (henceforth we assume positive nonlinearity, $\chi > 0$). We also scale the amplitude of the kth mode as $A_k = a_k \exp[i\alpha(t^2 - t_0^2)/2 + i\omega_k(t-t_0) + i\chi_1 \int_{t_0}^{t} |b(t')^2| dt'] \sqrt{\chi}/\alpha^{1/4}$ and the time as $\tau = \sqrt{\alpha} t$. Finally, we introduce the frequency mismatch $\Delta_k \equiv \omega_0 - \omega_k$, the scaled frequency mismatch $\tilde{\Delta}_k = \Delta_k/\sqrt{\alpha}$, and the scaled bandwidth of the continuum $\tilde{\Gamma} = \Gamma\sqrt{\alpha}$ and use the conserved quantity, $\sum_k |a_k|^2 + |b|^2 = 1$ to get

$$i\partial_t A_k + \frac{\chi_1}{\sqrt{\alpha}} A_k - \sum_{k'} |A_{k'}|^2 A_k + \tau A_k = \mu_k e^{-i\tilde{\Delta}_k(\tau-\tau_0)}, \quad (4)$$

where we defined the parameter $\mu_k = g_k\sqrt{\chi}\alpha^{3/4}$. We will ignore the second term $[\chi_1 A_k \sqrt{\alpha}]$ since it only shifts the time axis. Notice that in this general model, even if $\chi_2 = 0$ or if $\chi_1 = 0$, Eq. (4) holds—as long as the effective nonlinearity obeys $\chi > 0$. That is, even if just part of the system is nonlinear, the entire system will behave in a nonlinear fashion, owing to the coupling between the subsystems. This means that the source of the nonlinearity is unimportant for the derivation of Eq. (4). When examining Eq. (4), one notices that each mode crosses the linear resonance at a different time, according to its specific detuning from the resonance frequency ω_0. Also, each mode has a different source term, μ_k. Since all the modes are coupled, we will not study the dynamics of each mode separately. Instead, we will look for a superposition of the continuum modes that will effectively depend on a reduced number of parameters. First, we find an approximate solution for each mode when approaching the linear resonance. Since this happens before the continuum modes acquire any significant population, their evolution is still in the linear regime, and the nonlinear term of Eq. (4) can be neglected. By integrating Eq. (4) we get

$$\begin{aligned} A_k(\tau) &= -i\mu_k \int_{\tau_0}^{\tau} \exp\left[-i\tilde{\Delta}_k(\tau' - \tau_0) + \frac{1}{2} i(\tau^2 - \tau'^2)\right] d\tau' \\ &= -i\mu_k \sqrt{\frac{\pi}{2i}} \exp\left[\frac{i}{2}\left(\tilde{\Delta}_k^2 + \tau^2 + 2\tilde{\Delta}_k\tau_0\right)\right] \\ &\quad \times \left\{\Phi\left[\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}(\tilde{\Delta}_k + \tau)\right] - \Phi\left[\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}(\tilde{\Delta}_k + \tau_0)\right]\right\} \\ &\underset{\tau_0 \ll \tau < -\tilde{\Delta}_k}{\approx} \frac{\mu_k}{\tau} \exp[-i\tilde{\Delta}_k(\tau - \tau_0)] \end{aligned} \quad (5)$$

where $\Phi[x]$ is the error function. For autoresonant evolution, the phase of the mode must approach a constant value while approaching the linear resonance. Here, all the modes apart from the central mode ($\Delta_k = 0$) have oscillating phases when approaching the linear resonance. However, the approximate solution in Eq. (5) implies that choosing a proper superposition of continuum modes could yield a function experiencing phase locking. We choose a superposition of the modes A_k, scaled such that it will coincide with the case of two coupled discrete levels (for example, as in Ref. [7]), that is, a very narrow continuum [$\Gamma \to 0$ and $g_k^2 = \kappa^2\delta(\Delta_k)$]. The chosen superposition is

$$\sum_k \frac{g_k}{\kappa} A_k \approx \sum_k g_k \frac{\mu_k}{\kappa \tau} e^{-i\tilde{\Delta}_k(\tau-\tau_0)} = \frac{\mu}{\tau} e^{-\tilde{\Gamma}|\tau-\tau_0|}$$
$$= \frac{\mu}{\tau} e^{-\tilde{\Gamma}(\tau-\tau_0)}, \tag{6}$$

where we define $\mu = \frac{\sqrt{\chi}\kappa}{\alpha^{3/4}}$. We can now define a memory function $K(t) = \sum_k g_k^2 e^{i\Delta_k t}$, which, for the Lorentzian-shaped continuum, yields an exponentially decaying function $K(t) = \kappa^2 e^{-\Gamma|t|}$ [we remove the absolute value because $\tau \geqslant \tau_0$]. Notice that for $\Gamma \to 0$ our result converges to the simple case of discrete coupled states, and Eq. (4) describes the evolution of two coupled oscillators [15].

The expression described by Eq. (6) suggests an intriguing situation where proximity of the linear resonance forces the superposition of continuum modes to become a real function. That is, even though the amplitude of each continuum mode has a complex value, their superposition would nonetheless always attain a real value. This suggests that perhaps a collective autoresoance effect is taking place, where many continuum modes phase lock with the discrete state and are collectively being amplified by it. Without the collective effect, one cannot expect autoresonant phase locking for all the continuum modes, since for many of them the source amplitude is too weak to maintain the phase locking. But for those that do phase lock, we expect collective amplification. To maintain the resonance throughout evolution, we expect the nonlinearity to follow the chirp, that is, $\sum_k |A_k|^2 \approx \tau$. This means that, throughout evolution, more and more modes will phase lock and condensate together to a state with a single phase and growing amplitude. This result, as expressed by Eq. (6), does not imply that the amplified modes experience the phase locking stage and the amplification simultaneously, since the continuum modes do not cross the linear resonance exactly together. Actually, the resonance crossing and phase locking for modes with positive detuning ($\Delta_k > 0$) precedes the crossing of modes with negative detuning ($\Delta_k < 0$). We expect this asymmetry to manifest itself in the final occupation of the continuum modes, where the positively detuned modes are expected to have higher occupation. Also, since the coupling constant μ_k decays with the detuning, the modes at the tails of the Lorentzian will not be efficiently amplified. The most efficient amplification is a compromise between the time spent by the mode in resonant phase locking and the strength of the coupling constant. Modes closer to the center of the Lorentzian are coupled stronger to the source. On the other hand, such "central modes" cross the resonance later, thus spend less time being phase locked to the source, and therefore have less time to be amplified significantly. These arguments imply that we should expect a shift of the continuum peak towards the positively detuned frequencies. The width of the Lorentzain also plays a major role in the occupation of the modes of the continuum. A wider continuum (increased Γ) has a higher coupling constant at the tails, and can support efficient amplification of farther detuned modes. As such, efficient amplification will occur for a wider stripe of continuum modes; thus we expect the final distribution of continuum modes to be broader than the initial distribution.

The resonance-crossing stage is characterized by two parameters, which determine the evolution of the system. To see this, we multiply Eq. (4) by g_k, sum over all modes, and again neglect the weak nonlinearity of the early stage prior to resonance crossing. Defining $Y = \sum_k \frac{g_k}{\kappa} A_k e^{\tilde{\Gamma}(\tau-\tau_0)}$ transforms Eq. (4) into

$$i\partial_t Y - \sum_{k'} |A_{k'}|^2 Y + \tau Y - i\tilde{\Gamma} Y = \mu. \tag{7}$$

Therefore, the capture into resonance is mainly controlled by two parameters,μ, which is the equivalent of the threshold parameter when coupling a driving oscillator to a single driven oscillator, and $\tilde{\Gamma}$ that plays the role of damping. However, varying the other parameters (the chirp α and the nonlinearity χ), changes the details of the exact dynamics, and as such they can yield modifications to the evolution of each mode separately. We note that autoresonance between a driver and a single nonlinear driven oscillator was studied recently in the presence of damping and fluctuations [16]. However, here the source of damping and fluctuations are the continuum modes, which are in fact coupled driven systems. That is, for our case of coupling to a continuum of modes, one cannot separate the damping from the autoresonant evolution: without the coupling to the continuum there is no damping, but at the same time there is also no autoresonant evolution.

III. NUMERICAL RESULTS FOR THE SYSTEM

Up to this point, the system was treated analytically, under approximations. It is however important to simulate the actual dynamics of the system numerically. To do that we manipulate Eqs. (1a) and (1b). We define the amplitudes $\tilde{b} \equiv b \exp[i\omega_0 t + i\chi_2 \int_{t_0}^t \sum_{k'} |\tilde{a}_{k'}(\tau)|^2 d\tau]$ and $\tilde{a}_k \equiv a_k \exp[i\omega_k t + i\chi_2 \int_{t_0}^t \sum_{k'} |\tilde{a}_{k'}(\tau)|^2 d\tau]$, and we get two coupled equations describing the evolution of the continuum modes and of the discrete system:

$$\partial_t \tilde{b} = i\chi_2 \tilde{b} - i\chi|\tilde{b}|^2\tilde{b} - i\alpha t\tilde{b} - i\sum_k g_k \tilde{a}_k e^{i\Delta_k t}, \tag{8a}$$

$$\partial_t \tilde{a}_k = -i g_k \tilde{b} e^{-i\Delta_k t}, \tag{8b}$$

where we replaced $\sum_{k'} |\tilde{a}_{k'}|^2$ by $1 - |\tilde{b}|^2$. From now on, we neglect the term $i\chi_2\tilde{b}$ on the right-hand side (RHS) of Eq. (8a), since it only shifts the time $T_R = -\chi/\alpha$ at which the system crosses the (linear) resonance with the center of the continuum by the delay $\Delta T_R \approx \chi_2/\alpha$.

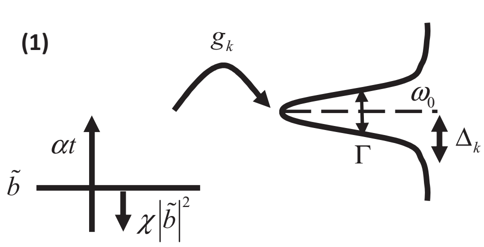

FIG. 1. Schematics of a discrete level $\tilde{b}$ coupled to a Lorentzian continuum. The natural frequency of the discrete level, ω_0, is shifted by a linear chirp and nonlinearity.

Equation (8a) describes the evolution of a driven nonlinear oscillator, where the driver resonance frequency is adiabatically varied. The terms $\chi|\tilde{b}|^2\tilde{b}$ and $i\alpha t\tilde{b}$ describe the nonlinearity of the driven oscillator and the chirped frequency of the driver. A diagrammatic representation of the system is shown in Fig. 1. The last term on the RHS of Eq. (8a), $\sum_k g_k\tilde{a}_k e^{i\Delta_k t}$, is the sum of the of the continuum modes with appropriate amplitudes and phases. The continuum modes have amplitudes and phases which are both time dependent, and the dynamics of each mode is governed by Eq. (8b). This term acts as the driving source for the oscillations of the amplitude and phase of the driven oscillator, $\tilde{b}$. In analogy with autoresonance dynamics, we expect that, if possible, the phase $\tilde{b}$ will lock with the phase of the source term. (This prediction is exactly the same as the one made above, when we examined the system from the continuum point of view, and concluded that $\sum_k \frac{g_k}{\kappa}A_k$ should phase lock with the discrete state). However, unlike the case of a driven nonlinear oscillator coupled to a single oscillator, here the source is a continuous set of oscillators, each with its own frequency and time-dependent amplitude. Also, each continuum mode is effectively coupled to all the other modes through their coupling with $\tilde{b}$, which introduces feedback into the dynamics.

We will now transform Eqs. (8a) and (8b) to a single integro-differential equation to eliminate the continuum. By integrating Eq. (8b) from the beginning of the process at t_0, we get the kth amplitude at the time t:

$$\tilde{a}_k(t) = -ig_k\int_{t_0}^{t}\tilde{b}(t')e^{-i\Delta_k t'}dt' + \tilde{a}_k(t_0). \tag{9}$$

Substituting Eq. (9) into Eq. (8a), we find that $\tilde{b}$ evolves according to

$$\partial_t\tilde{b} = -\int_{t_0}^{t}K(t-t')\tilde{b}(t')dt' - i\chi|\tilde{b}|^2\tilde{b} - i\alpha t\tilde{b} - if(t), \tag{10}$$

where we defined the memory function

$$K(t-t') = \sum_k g_k^2 e^{i\Delta_k(t-t')}, \tag{11a}$$

and a fluctuations term (Langevin force)

$$f(t) = \sum_k g_k\tilde{a}_k(t_0)e^{i\Delta_k t}, \tag{11b}$$

which depends on the amplitude of the continuum modes at t_0. For the Lorentzian-shaped continuum, the memory function decays exponentially, $K(t-t') = \kappa^2 e^{-\Gamma|t-t'|}$. From now on, we will assume that the continuum is initially empty, that is, the initial amplitudes of the continuum modes are all zero [$a_k(t_0) = 0,\ \forall k$], which means that the fluctuations term vanishes.

Next, we separate $\tilde{b}(t)$ into amplitude and phase by defining $\tilde{b}(t) \equiv x(t)e^{i\theta(t)}$, and separating Eq. (10) into real and imaginary parts:

$$\partial_t x = -\int_{t_0}^{t}x(t')\mathrm{Re}\left\{K(t-t')e^{i[\theta(t')-\theta(t)]}\right\}dt', \tag{12a}$$

$$\partial_t\theta = -\frac{1}{x}\int_{t_0}^{t}x(t')\mathrm{Im}\left\{K(t-t')e^{i[\theta(t')-\theta(t)]}\right\}dt' - \chi x^2 - \alpha t. \tag{12b}$$

Equations (12a) and (12b) are similar to the equation describing the dynamics of a single driven oscillator with damping. However, here the driving force comes from the occupied modes in the continuum, while at the same time the occupied modes are responsible for the damping of the driven oscillator. We are interested in a continuum which is narrow enough so that the Markovian approximation is inapplicable, and the dynamics, in general, has memory. We begin by solving numerically the evolution of x and θ for parameters above the autoresonance threshold. Figure 2(a) shows an example of efficient amplification of the continuum modes. The red line shows the decay of $x^2 = |\tilde{b}|^2$ (population of the discrete state), the black-dashed line displays the amplification of $\sum_k|\tilde{a}_k|^2$, and the blue circles represent the analytic result $-\alpha t/\chi$. Indeed, the simulation shows that the continuum modes are amplified at the expense of the discrete mode, which in turn decays linearly in time (as we have found analytically). We also plot the dynamics of $|\sum_k g_k\tilde{a}_k e^{i\Delta_k t}|^2$ (green dotted dashed). As expected, this superposition grows in time, since all the amplified modes comprising it are phase locked; hence they always interfere constructively. The blue line in the center of Fig. 2(b) displays the simulated evolution of the phase mismatch between $\sum_k g_k\tilde{a}_k e^{i\Delta_k t}$ and $\tilde{b}(t)$. As expected, this superposition phase locks with the discrete mode. Also, we plot the phase mismatch between $\tilde{a}_k(t)e^{i\Delta_k t}$ and $\tilde{b}(t)$ for five different values of $\Delta k/\Gamma$. As shown, the strongly detuned modes (for both positive and negative detuning, $\Delta k/\Gamma = \pm 25$) do not phase lock [17]. On the other hand, the modes at the vicinity of the center of the continuum do phase lock and are amplified [exactly as was predicted by Eq. (5)]. For comparison, we plot the dynamics of x^2 and of the phase mismatch below the autoresonance threshold [Figs. 2(c) and 2(d), respectively], where we only change κ to 0.005. Now, the discrete level remains populated and the phases do not lock. Notice that for these parameters the system is very different from the simpler problem of two coupled oscillators. Namely, describing the system as two coupled oscillators requires that the memory time of the system, $1/\Gamma$, will be much longer than the typical time scale for variations in $\tilde{b}$. However, for autoresonant dynamics such typical time scale is χ/α, which is comparable to the memory time. For the same reason, the dynamics below the autoresonant threshold is not Markovian. This is because for Markovian evolution, the memory time must be much shorter than the typical variation time of $\tilde{b}$, whereas here the memory function resembles a delta function in time, $K(t-t') = \frac{\kappa^2}{\Gamma}\delta(t-t')$, transforming Eq. (10a) into

$$\partial_t x = -\frac{\kappa^2}{\Gamma}x(t) \tag{13}$$

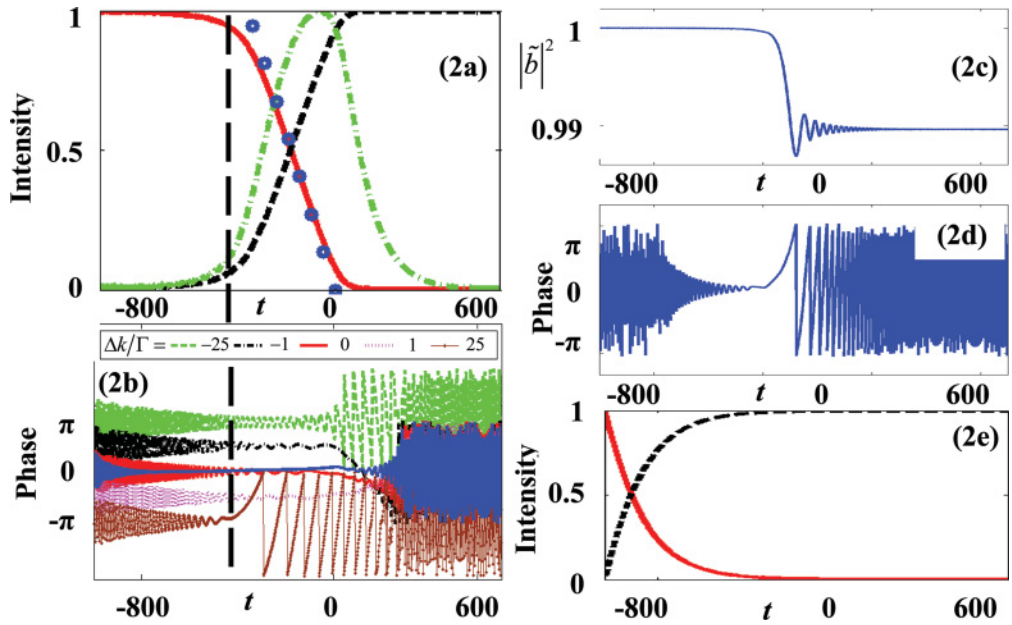

FIG. 2. (Color online) (a) Autoresonant dynamics for $\alpha = 0.0015$, $\chi = 0.5$, $\kappa = 0.035$, and $\Gamma = 0.005$. The red-solid line represents $|\tilde{b}|^2 = x^2$, the black-dashed line represents $\sum_k |\tilde{a}_k|^2$, the blue circles mark the analytic result for autoresonant phase locking ($|\tilde{b}|^2 = -\alpha t/\chi$), and the green-dashed-dotted line shows the evolution of $|\sum_k g_k \tilde{a}_k e^{i\Delta_k t}|^2$ (normalized to unity). Clearly, the discrete level decays linearly (in time) into the continuum. The vertical black-dashed line marks the crossing of the linear resonance. (b) Evolution of the phase mismatch between $\tilde{b}$ and $\sum_k g_k \tilde{a}_k e^{i\Delta_k t}$ (central blue line), and of between $\tilde{b}$ and different continuum modes (other lines), both positively and negatively detuned (for clarity, the plots are slightly shifted from one another). Dynamics (c) of $|\tilde{b}|^2$ and (d) of the phase mismatch between $\tilde{b}$ and $\sum_k g_k \tilde{a}_k e^{i\Delta_k t}$, below the threshold for $\alpha = 0.0015$, $\chi = 0.5$, $\kappa = 0.005$, and $\Gamma = 0.005$. (e) Markovian exponential decay ($|\tilde{b}|^2$, solid-red line) and amplification of a broadband continuum ($\sum_k |\tilde{a}_k|^2$, dashed-black line) under $\alpha = 0.0015$, $\chi = 0.5$, $\kappa = 0.07$, and $\Gamma = 1$.

for which the solution is an exponential decay of x. Here, unlike the autoresonant evolution, the system decays exponentially before crossing the linear resonance. In this regime, there is no nonlinear phase locking with the continuum, and no fixed phase relation between the various continuum modes. Figure 2(e) shows the exponential decay and amplification of x^2 and $\sum_k |\tilde{a}_k|^2$, respectively (red and black, respectively) for $\alpha = 0.0015$, $\chi = 0.5$, $\kappa = 0.07$, and $\Gamma = 1$. Henceforth throughout the rest of the paper, we will use parameters that we are far from the exponential decay regime, so that we can indeed observe non-Markovian autoresonant decay.

IV. AUTORESONANT THESHOLD

We now turn to study the threshold for autoresonant decay. Figure 3 displays the final population of the discrete level, x_f^2, as a function of μ for $\alpha = 0.0015$, $\chi = 0.5$. The black-dashed-dotted, blue-solid, red-dashed, and green-dotted lines represent $\Gamma = 0.0001, 0.002, 0.005, 0.01$, respectively. For $\Gamma = 0.0001$, the continuum is narrow enough to be considered a discrete level. The damping for the narrow band is negligible, and the dynamics is identical to that of the two coupled oscillators system, where indeed the threshold for the transition to autoresonant decay is sharp, occurring for $\mu \approx 0.4$. When the width of the continuum is increased, the transition to autoresonant decay is not sharp anymore. Basically, increasing the continuum width makes the discrete state couple to more continuum modes. The phase oscillations of these modes makes it harder for them to phase lock, and stronger coupling (larger μ) is required for complete population inversion.

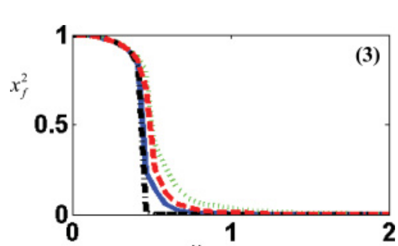

FIG. 3. (Color online) Final occupation of the discrete level, x_f^2, vs μ for $\alpha = 0.0015$, $\chi = 0.5$, and $\Gamma = 0.0001$ (black dashed dotted), 0.002 (blue solid), 0.005 (red dashed), and 0.01 (green dotted).

V. EMISSION SPECTRUM ANALYSIS

It is interesting to study the properties of the occupied continuum modes, as revealed by the simulation. We find that indeed the width of the emitted spectrum increases when the coupling constant and/or Γ are increased. To show that, we first calculate the evolution of the continuum modes by integrating Eq. (8b):

$$\tilde{a}_k(t) = -i g_k \int_{t_0}^{t} \tilde{b}(t') e^{-i\Delta_k t'} dt'. \tag{14}$$

We define the final distribution of the continuum modes as $\tilde{a}_k^f$, and evaluate the continuum width by calculating

$$\Delta_k^f = \left(\int_{-\Lambda}^{\Lambda} |a_k^f|^2 \Delta_k^2 d\Delta_k \right)^{1/2} \Big/ \left(\int_{-\Lambda}^{\Lambda} |a_k^f|^2 d\Delta_k \right)^{1/2}, \tag{15}$$

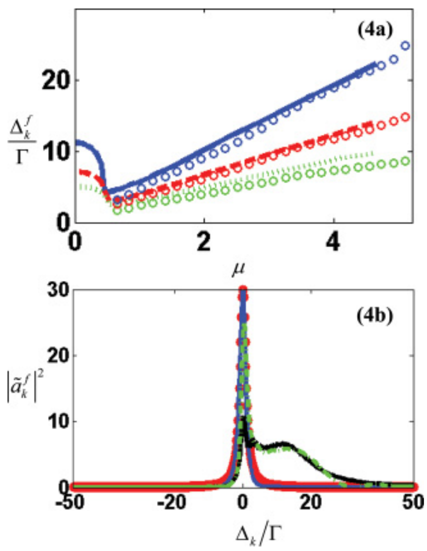

FIG. 4. (Color online) (a)Width of the distribution of the continuum modes, Δ_k^f, vs μ, for $\alpha = 0.0015$, $\chi = 0.5$, and $\Gamma = 0.002$ (blue solid), 0.005 (red dashed), and 0.01 (green dotted). The circles mark the analytic model. (b) Final distribution of the continuum modes, $|\tilde{a}_k^f|^2$, vs Δ_k/Γ. The blue (black) line represents the final distribution of the continuum modes below (above) the threshold for $\kappa = 0.01$ ($\kappa = 0.08$) and for $\alpha = 0.0015$, $\chi = 0.5$, and $\Gamma = 0.005$. The red line mark the natural shape of the coupling to the continuum. The green-dashed-dotted line marks the semianalytical calculation of the final distribution.

where $\Lambda \gg \Gamma$ is a cutoff frequency for the integral. We plot in Fig. 4(a) the width normalized to Γ vs μ for $\alpha = 0.0015$, $\chi = 0.5$. μ is varied by varying κ. The blue, red, and green lines represent $\Gamma = 0.002, 0.005, 0.01$, respectively. We study the dynamics of the continuum as a function of κ and Γ. First, we see that increasing κ indeed increases the width of the distribution of populated modes in the continuum. Notice that the width goes beyond the width associated with weak coupling (below the autoresonance threshold). Also, there is a clear autoresonant transition: below the threshold the "continuum width" barely changes while increasing the coupling, whereas above the threshold the width grows monotonically with increasing coupling strength. We also find, as expected, that the continuum width increases when increasing Γ. This is not seen in Fig. 4(a), since the width plotted there is normalized to Γ. Figure 4(b) shows the final distribution of the continuum modes below and above the threshold (blue and black lines, respectively) for $\kappa = 0.01$ ($\kappa = 0.08$) and $\alpha = 0.0015$, $\chi = 0.5$, and $\Gamma = 0.005$. As expected, the profile changes drastically. Below threshold, the continuum distribution is nearly symmetric and has a shape very close to the line shape of the original continuum ($a_k^f \propto g_k$), marked here with red line. Above the threshold, the distribution is shifted to the positive detuned frequencies, and has a wider width. We emphasize that the exact dynamics also depends on α and χ. That is, the width grows at a different rate as a function of μ, for different chirp rates or nonlinearities. However, the main results do not change. There is always a clear transition to autoresonant dynamics, where the width of the spectrum of the continuum modes always increases.

To obtain an approximate analytic expression for the final occupation of the continuum modes, one first needs to understand the dynamics of the system. The dynamics of each continuum mode is dictated by Eq. (14). We expect that the most amplified modes are those that compromise between phase matching with $\tilde{b}$ for the longest time and having a strong coupling to the continuum (as mentioned above, this compromise yields positive detuning in the final modal occupation). However, the major problem here is that, unlike the autoresonant coupling process between a single driven mode and a driver with a known, externally controlled phase, for coupling to a continuum of modes the phase of $\tilde{b}$ is dynamically changing, according to the evolution of the system, and it is basically unknown. What renders things even harder is the memory of the system, which makes the phase at any given time depend on its past values. We overcome these problems by making some reasonable assumptions that yield a semianalytic solution for the continuum width. First, we assume that below the resonance time ($T_R = -\chi/\alpha$) the amplitude $\tilde{b}$ does not decrease, but only its phase oscillates. This is of course inaccurate, since the coupling to the continuum, represented by the expression $i\sum_k g_k \tilde{a}_k e^{i\Delta_k t}$ in Eq. (8a) has both real and imaginary parts. The imaginary part represents a dynamic level shift of the discrete level, resulting from the coupling to the continuum, while the real part represents dynamic tunneling to the continuum, which results in the decay of $\tilde{b}$. Our assumption becomes more accurate for narrower continua and weaker coupling constants, where the tunneling is negligible as long as the discrete level is slightly detuned from the continuum peak. The most significant oscillations in this regime arise from the chirp and the nonlinearity, $\theta^{(0)} = -\frac{\alpha}{2}t^2 - \chi t$. The correction to the phase oscillations in this regime comes from the dynamic level shift which, to first order in κ^2 (the small parameter in the problem as long as κ^2 is small enough), can be approximated using Eq. (12b) as

$$\partial_t \theta^{(1)} = -\int_{-\infty}^{t} \mathrm{Im}\left(K(t-t')\exp\left\{i\left[\frac{\alpha}{2}(t'^2 - t^2) + \chi(t'-t)\right]\right\}\right) dt'$$
$$= i\exp\left[\frac{i}{2\alpha}(\alpha t + \chi + i\Gamma)^2\right]\frac{\sqrt{\pi}}{2\alpha}\left\{\sqrt{2i\alpha} + e^{i\pi/4}\sqrt{\frac{\alpha}{2}}\Phi\left(\frac{e^{i\pi/4}[\alpha t + \chi + i\Gamma]}{\sqrt{2\alpha}}\right)\right\},\ t < T_R. \quad (16)$$

Here, we assumed that the initial time of the process is $t \to -\infty$, but the results do not change significantly for a finite initial time. Next, we assume that above T_R, phase locking takes place and the amplitude decreases as $\tilde{b} = \sqrt{-\alpha t/\chi}e^{i\theta^{(1)}(t)}$. Here $\theta^{(1)}(t)$ is an unknown phase, of order κ^2, which we would like to estimate since, as mentioned above, it dictates what modes are amplified. The dynamics of the phase in this regime is determined by Eq. (12b) with phase locking:

$$\partial_t\theta^{(1)} = -\frac{1}{x}\int_{t_0}^{t} x(t')\mathrm{Im}\left\{K(t-t')e^{i[\theta(t')-\theta(t)]}\right\}dt',\ t \geqslant T_R. \tag{17}$$

Notice that at the vicinity of $t = 0$ the integral approximately vanishes, since $x(t \to 0)$ decays to zero. This means that at $t = 0$ the discrete level is resonant with the mode $\Delta_k = 0$, while below $t = 0$ it is resonant with positively detuned modes. We find numerically that $\partial_t\theta^{(1)}$ indeed goes to zero for $t \to 0$, with a moderate slope. We also find that it can be satisfactory fitted with a Taylor series expanded up to a third order. We therefore do not explicitly calculate the integral in (17). Instead, we fit it with a Taylor series around $t = 0$, and by assuming continuity of the phase and its derivatives at T_R, we find the coefficients of the series up to the third order. In doing that, we use the Taylor expansion of the RHS of Eq. (16) to find the evolution of the phase $\theta^{(1)}$ for $t \geqslant T_R$, and by integrating numerically $\tilde{b}$ from T_R to $t = 0$ we get our approximate solution. The circles in Fig. 4(a) show the calculated widths for the appropriate parameters. We find good agreement between the approximate analytic model and the numerical results. As mentioned above, for higher continuum widths or higher coupling constants the numerical results deviate from the analytic model. Nonetheless, the model does capture the main results of the simulations. Finally, the green line in Fig. 4(b) shows the calculated distribution of continuum modes for the same parameters as those of the black line. As shown there, the calculated profiles are very similar. A more accurate model should take into consideration the exact transition to autoresonant dynamics (which is not as sharp as our analytic model suggests), the decay to the continuum before the nonlinear capture into resonance (which influences the amplitude and phase of the discrete state before the phase locking regime), and the exact dynamics during the autoresonant evolution.

VI. CONCLUSION

To summarize, we have shown that a discrete level can be captured into a continuous resonance with a continuum set of modes. We have found that the final distribution of the continuum modes can be increased compared to the natural width of the continuum. Also, at the end of the process ($t \sim 0$), all the phases are locked. The combination of phase coherence between the different modes and a broad spectrum can lead to emission of short pulses, for example, an atom emitting a transform-limited photon with narrow temporal width. These ideas can be extended to the quantum regime, where the problem presented here is equivalent to an atom, modeled as a quantum oscillator initially in a coherent state, coupled to a narrow Lorentzian-shaped continuum of coherent states. However, if the atom is not in a coherent state, or if the atom is not modeled as an oscillator, quantum fluctuations may have an effect on the decay process. Also, different types of continua will yield different memory functions and therefore different results. For example, emission next to a band edge of a photonic crystal will surely behave differently, due to the strong non-Markovian dynamics, and effects, like superradiance at the vicinity of the band edge, would change due to the predetermined autoresonant evolution [18]. For example, it will be interesting to design autoresonant decay that will yield stronger superradiant intensities. Finally, it will be very interesting to study how fluctuations in the continuum will affect the dynamics, for example, by assuming that initially $\tilde{a}_k(t_0) \neq 0$, and that the modes are distributed in a coherent fashion, or incoherently (e.g., thermal distribution). In such a case, the fluctuations term $f(t)$ is not zero, which introduces fluctuations to the system. Is it still possible to phase lock with the continuum modes? Will autoresonance overcome random fluctuations of the continuum? Would the reverse process be possible? That is, can we begin with the discrete level empty, and use autoresonance to phase lock it to the fluctuating continuum, such that eventually the discrete level will amplify? These and related questions offer much thought for future work.

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[17] Note that from $t \approx 0$ and on, the amplitude of the discrete state vanishes, and there is nothing that can maintain the phase locking among the continuum modes; hence, the continuum modes go out of phase with one another, as manifested by the large oscillations after $t > 300$ in Fig. 2(b)

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